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Brief paper

# Stochastic consensus seeking with communication delays ☆

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ABSTRACT

This paper investigates the consensus problem of dynamical networks of multi-agents where each agent can only obtain noisy and delayed measurements of the states of its neighbors due to environmental uncertainties and communication delays. We consider general networks with fixed topology and with switching (dynamically changing) topology, propose consensus protocols that take into account both the noisy measurements and the communication time-delays, and study mean square average-consensus for multi-agent systems networked in an uncertain environment and with uniform communication time-varying delays. Using tools from differential equations and stochastic calculus, together with results from matrix theory and algebraic graph theory, we establish sufficient conditions under which the proposed consensus protocols lead to mean square average-consensus. Simulations are also provided to demonstrate the theoretical results.



## 1. Introduction

While consensus problems have a long history in both computer science (Lynch, 1997) and statistics (DeGroot, 1974), there has been a recent surge of interests among various disciplines of engineering and science in problems related to networked systems of multi-agents that emphasize consensus or agreement (see, e.g., Bauso, Giarre, & Pesenti, 2009; Bliman & Ferrari-Trecate, 2008; Cao, Morse, & Anderson, 2008a,b; Carli, Fagnani, Speranzon, & Zampieri, 2006; Fax & Murray, 2004; Hatano & Mesbahi, 2005; Huang & Manton, 2009, 2010a,b; Jadbabaie, Lin, & Morse, 2003; Li & Zhang, 2009; Moreau, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Ren, Beard, & Kingston, 2005; Xiao, Boyd, & Kim, 2007; also see Olfati-Saber, Fax, & Murray, 2007 for a recent survey and extensive references therein). Consensus problems naturally arise when a group of agents, often distributed over a network, are seeking agreement upon a certain quantity of interest, which might be attitude, position, velocity, voltage, direction, temperature, and so on, depending on different applications.

Networked systems are often subject to environmental uncertainties and communication delays, which make it difficult or impossible for a networked agent to obtain timely and accurate information of its neighbors. Moreover, link gains/failures and formation reconfiguration make it necessary to address consensus problems for networks with switching network topology. The recent work of Huang and Manton (2009, 2010a,b) studies stochastic consensus problems of networked agents, with or without switching topology, in the discrete-time setting using algorithms from stochastic approximation. In Li and Zhang (2009), the work of Huang and Manton (2009) is extended to the continuous-time setting, and both necessary and sufficient conditions for stochastic consensus have been obtained for networks that are both balanced and containing a spanning tree (equivalent to strongly connected and balanced; Li & Zhang, 2009). Other work on consensus problems that explicitly takes into account measurement and environmental noises in different contexts includes (Bauso et al., 2009; Borkar & Varaiya, 1982; Carli et al., 2006; de Castro & Paganini, 2004; Hatano & Mesbahi, 2005; Ren et al., 2005; Tsitsiklis & Athans, 1984; Tsitsiklis, Bertsekas, & Athans, 1986; Xiao et al., 2007), in some of which the noises are modeled as deterministic but unknown disturbances (e.g., Bauso et al., 2009; de Castro & Paganini, 2004). On the other hand, consensus problems with communication delays have also been studied extensively in recent years (see, e.g., Bliman & Ferrari-Trecate, 2008; Lin & Jia, 2009; Lu, Ho, &

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Kurths, 2009; Munz, Papachristodoulou, & Allgower, 2010; Olfati-Saber & Murray, 2004; Tian & Liu, 2008; Wang & Slotine, 2006; Xiao & Wang, 2008; Yang & Fang, 2010; Zhu & Cheng, 2010). None of the above mentioned work, however, has investigated stochastic consensus problems of networks with communication delays, either in a discrete- or continuous-time setting, while delays are ubiquitous in communication networks.

The purpose and main contribution of this paper is to investigate stochastic consensus problems with communication time-delays. Following the time-varying consensus protocol introduced by Huang and Manton (2009) for discrete-time systems and by Li and Zhang (2009) for continuous-time systems, both without communication delays, we propose a time-varying consensus protocol that takes into account both the measurement noises and general time-varying communication time-delays. We take a continuous-time approach using differential equations and stochastic calculus, and aim to provide conditions under which the proposed consensus protocol actually leads to consensus for networks with strongly connected and balanced topology. Moreover, the consensus results are extended to networks with arbitrary deterministic switching topology and with Markovian random switching topology. Explicit delay upper bounds for guaranteeing consensus are obtained in each case.

The rest of this paper is organized as follows. In Section 2, we formulate the consensus problem, propose the consensus protocol, and introduce two notions of stochastic consensus. The main consensus results are presented in Section 3, followed by demonstrations through numerical simulations in Section 4. The detailed proofs for the main results are included in Section 5, where a Gronwall–Bellman–Halanay type inequality is established, which plays an essential role in proving the consensus results. The paper is concluded by Section 6, which also points out some possible future research along the line of this paper.

## 2. Problem formulation and consensus protocols

### 2.1. Network topology

The interaction topology of a network of $n$-agents is modeled by a weighted digraph (or directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order $n$ with set of nodes $\mathcal{V} = \{v_1, \ldots, v_n\}$, set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and a weighted adjacent matrix $\mathcal{A} = [a_{ij}]_{n\times n}$ with nonnegative elements $a_{ij}$. An edge of $\mathcal{G}$ is denoted by $e_{ji} = (v_j, v_i)$. An edge $e_{ji}$ exists if and only if $a_{ij} > 0$. It is assumed that $a_{ii} = 0$ for $i = 1, \ldots, n$. The set of neighbors of a node $v_i$ is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Let $x_i \in \mathbb{R}$ denote the value of node $v_i$, which is a scalar quantity of interest. Denote the set $\{1, \ldots, n\}$ by $\mathcal{I}$. The *graph Laplacian* $\mathcal{L}$ of the network is defined by

$$\mathcal{L} = D - \mathcal{A}, \tag{2.1}$$

where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is the degree matrix of $\mathcal{G}$ with elements $d_i = \sum_{j\neq i} a_{ij}$ and $\mathcal{A}$ is the weighted adjacent matrix. A digraph (and the corresponding network) is *strongly connected* if there is a directed path connecting any two arbitrary nodes in the graph. A digraph (and the corresponding network) is said to be *balanced* if $\sum_{j\neq i} a_{ij} = \sum_{j\neq i} a_{ji}$ for all $i \in \mathcal{I}$.

### 2.2. Consensus protocols

Consider each node of the graph to be a dynamic agent with dynamics

$$\dot{x}_i = u_i, \quad i \in \mathcal{I}, \tag{2.2}$$

where the state feedback $u_i = u_i(x_{i_1}, \ldots, x_{i_k})$ is called a *protocol* with topology $\mathcal{G}$ if the set of nodes $\{x_{i_1}, \ldots, x_{i_k}\}$ are all taken from the set $\{v_i\} \cup \mathcal{N}_i$, i.e. only the information of $v_i$ itself and its neighbors are available in forming the state feedback for the node $v_i$.

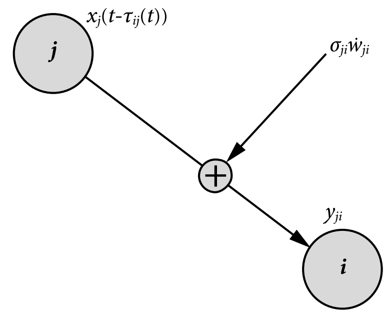


**Fig. 1.** Delayed measurements with additive noises.

We consider the following consensus protocol by Olfati-Saber and Murray (2004)

$$u_i = \sum_{v_j\in\mathcal{N}_i} a_{ij}(x_j - x_i), \quad i \in \mathcal{I}. \tag{2.3}$$

The above protocol requires that agent $i$ can obtain information from its neighbors in $\mathcal{N}_i$ timely and accurately, i.e. it assumes zero communication time-delay and accurate information exchange among agents. Let $y_{ji}$ be a measurement of $x_j$ by $x_i$ given by

$$y_{ji} = x_j + \sigma_{ji}\dot{w}_{ji}(t), \quad i \in \mathcal{I}, \tag{2.4}$$

where $\{\dot{w}_{ji}(t) : i, j = 1, \ldots, n\}$ are independent standard white noises and $\sigma_{ji} \geq 0$ represent the noise intensity. Replacing $x_j$ in (2.3) with the noisy measurement $y_{ji}$ gives the following stochastic consensus protocol

$$u_i = \sum_{v_j\in\mathcal{N}_i} a_{ij}(y_{ji} - x_i), \quad i \in \mathcal{I}. \tag{2.5}$$

If, in addition, time-varying communication delays are considered, we propose the following delayed stochastic consensus protocol

$$u_i(t) = c(t) \sum_{v_j\in\mathcal{N}_i} a_{ij}[y_{ji} - x_i(t - \tau_{ij}(t))], \quad i \in \mathcal{I} \tag{2.6}$$

where

$$y_{ji} = x_j(t - \tau_{ij}(t)) + \sigma_{ji}\dot{w}_{ji}(t), \quad i \in \mathcal{I},$$

and the time-varying delays $\tau_{ij}(t)$ lie in $[0, \tau]$ for some $\tau > 0$ and are assumed to be continuous in $t$. The function $c : \mathbb{R}^+ \mapsto \mathbb{R}^+$ in (2.6) is a piecewise continuous function satisfying

$$\int_0^\infty c(s)ds = \infty \quad \text{and} \quad \int_0^\infty c^2(s)ds < \infty. \tag{2.7}$$

The role of the function $c(t)$ is to attenuate the noise effects as $t \to \infty$. Condition (2.7), on the one hand, implies that $c(t)$ is vanishing as $t \to \infty$, but, on the other hand, not too fast due to $\int_0^\infty c(s)ds = \infty$. Without loss of generality, we can assume that $\sup_{t\geq 0} c(t) \leq 1$. If $\mathcal{N}_i$ is fixed, (2.6) gives a *fixed topology protocol*. If $\mathcal{N}_i$ is time-varying, we have a *switching topology protocol*. The communication delays and noisy measurements in the protocol (2.6) are illustrated by Fig. 1.

In this paper, we shall focus on the case when the time-varying delays are uniform, i.e. $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$.

### 2.3. Network dynamics

If the time-delays are uniform, i.e. $\tau_{ij}(t) = \tau(t)$ for all $i, j \in \mathcal{I}$, the collective dynamics of system (2.2) under the consensus protocol (2.6) can be written in a compact form of a stochastic delay differential equation (SDDE) as

$$dx(t) = c(t)[-\mathcal{L}x(t - \tau(t))dt + \Theta dW(t)], \tag{2.8}$$

where $W(t)$ is an $n^2$-dimensional standard Wiener process; $\mathcal{L}$ is the graph Laplacian of the network; and $\Theta \in \mathbb{R}^{n\times n^2}$ is a constant matrix defined by $\Theta = \operatorname{diag}(\Theta_1, \ldots, \Theta_n)$, where $\Theta_i$ is an $n$-dimensional row vector given by $\Theta_i = [a_{i1}\sigma_{1i}\ a_{i2}\sigma_{2i}\ \cdots\ a_{in}\sigma_{ni}]$.

### 2.4. Consensus notion

We introduce the following notion of mean square average-consensus for the multi-agent systems (2.2) under the consensus protocol (2.6) in an uncertain environment.

**Definition 2.1** (*Li and Zhang (2009)*). The agents in (2.2) are said to reach *mean square average-consensus* if $E|x_i(t)|^2 < \infty$ for all $t \geq 0$ and $i \in \mathcal{I}$ and there exists a random variable $x^*$ such that $E(x^*) = \text{avg}(x(0)) = \sum_{i=1}^{n} x_i(0)/n$ and $\lim_{t\to\infty} E|x_i(t) - x^*|^2 = 0$ for all $i \in \mathcal{I}$.

Huang and Manton (2009) defined and investigated both mean square consensus and almost sure consensus (called strong consensus) in the discrete-time setting, without emphasizing average-consensus and considering communication delays. Continuous-time mean square average-consensus has been defined and studied by Li and Zhang (2009), without considering communication delays.

## 3. Consensus results

In this section, we analyze the consensus properties of the dynamics of system (2.2).

### 3.1. Networks with fixed topology

We start by analyzing networks with fixed topology, i.e. the weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is time-invariant.

**Theorem 3.1.** *Suppose that $\mathcal{G}$ is a strongly connected and balanced digraph with $\mathcal{L}$ as its Laplacian. Let $\lambda_2(\hat{\mathcal{L}})$ denote the second smallest eigenvalue of $\hat{\mathcal{L}} = (\mathcal{L} + \mathcal{L}^T)/2$. If*

$$\tau < \frac{\lambda_2(\hat{\mathcal{L}})}{\|\mathcal{L}^2\|}, \tag{3.1}$$

*where $\tau > 0$ is an upper bound for the uniform time-varying delay $\tau(t)$ and $\|\cdot\|$ denotes the spectral norm, then the consensus protocol (2.6) leads to mean square average-consensus for the agents in (2.2).*

To prove this theorem, we introduce a so-called displacement vector as in Li and Zhang (2009) and Olfati-Saber and Murray (2004)

$$\delta(t) = x(t) - \mathbf{1}\alpha(t) = (I - J)x(t), \tag{3.2}$$

where $\mathbf{1}$ stands for the column $n$-vector with all ones, $\alpha(t) = \text{avg}(x(t)) = \frac{1}{n}\mathbf{1}^T x(t) = \frac{1}{n}\sum_{i=1}^{n} x_i(t)$, $I$ is the $n \times n$ identity matrix, and $J = \frac{1}{n}\mathbf{1}\mathbf{1}^T$. It is easy to see that

$$\mathbf{1}^T\delta(t) = \sum_{i=1}^{n} x_i(t) - n\alpha(t) = 0, \quad t \geq 0. \tag{3.3}$$

The dynamics of $\delta(t)$ are given by

$$d\delta(t) = c(t)[-\mathcal{L}\delta(t - \tau(t))dt + (I - J)\Theta dW(t)], \tag{3.4}$$

where we have used the fact that both $\mathbf{1}^T\mathcal{L}$ and $\mathcal{L}\mathbf{1}$ are zero vectors. The consensus analysis relies on the Lyapunov function candidate $V(t) = \delta^T(t)\delta(t) = |\delta(t)|^2, \quad t \geq 0.$

The quantity $\lambda_2(\hat{\mathcal{L}})$, called the *algebraic connectivity* of the graph $\mathcal{G}$, was originally introduced by Fiedler (1973) for undirected graphs and later extended to digraphs by Olfati-Saber and Murray (2004). The following property of the graph Laplacian $\mathcal{L}$ for strongly connected and balanced digraphs (see Theorem 7 of Olfati-Saber & Murray, 2004),

$$\delta^T\mathcal{L}\delta \geq \lambda_2(\hat{\mathcal{L}})|\delta|^2, \quad \forall 1^T\delta = 0, \tag{3.5}$$

plays an important role in ensuring that the protocol (2.6) leads to consensus of the agents in (2.2): The detailed proof for Theorem 3.1 is included in Section 5.

### 3.2. Networks with arbitrarily switching topology

In general, the network topology specified by the weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ can be time-varying due to node and link failures/creations, packet-loss, asynchronous consensus, formation reconfiguration, evolution, and flocking as pointed out by Olfati-Saber et al. (2007). To effectively model the dynamic changing of the network structures, we consider a collection of digraphs and introduce a general time-dependent switching signal, either deterministic or stochastic, to switch the network structures among the collection of digraphs.

We consider deterministic time-dependent switching in this subsection. Let $\mathcal{P}$ denote a finite index set and $\{\mathcal{G}_p : p \in \mathcal{P}\}$ a family of digraphs. Let $\sigma : \mathbb{R}^+ \to \mathcal{P}$ be a piecewise constant and right-continuous function called a *switching signal*. The collective dynamics (2.8) can be written as a switched system

$$dx(t) = c(t)[-\mathcal{L}_\sigma x(t - \tau(t))dt + \Theta dW(t)], \tag{3.6}$$

where $\mathcal{L}_p$ $(p \in \mathcal{P})$ is the corresponding graph Laplacian of $\mathcal{G}_p$.

**Theorem 3.2.** *Suppose that each $\mathcal{G}_p$ in $\{\mathcal{G}_p : p \in \mathcal{P}\}$ is a strongly connected and balanced digraph with $\mathcal{L}_p$ as its Laplacian. Let $\lambda_2(\hat{\mathcal{L}}_p)$ denote the second smallest eigenvalue of $\hat{\mathcal{L}}_p = (\mathcal{L}_p + \mathcal{L}_p^T)/2$. If*

$$\tau < \frac{\min_{p\in\mathcal{P}} \lambda_2(\hat{\mathcal{L}}_p)}{\max_{p\in\mathcal{P}} \|\mathcal{L}_p\|^2}, \tag{3.7}$$

*then the consensus protocol (2.6) leads to mean square average-consensus for the agents in (2.2) under any arbitrary deterministic switching signals.*

### 3.3. Networks with Markovian switching topology

Hybrid systems driven by continuous-time Markov chains have long been used to model many practical systems where abrupt changes in their structures and parameters caused by phenomena such as component failures and repairs as pointed out in Mao and Yuan (2006). In this subsection, we consider the case where the switching signal is modeled by a continuous-time Markov chain. More specifically, let $\sigma : \mathbb{R}^+ \to \mathcal{P}$ be a right-continuous Markov chain with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P(\sigma(t + \Delta) = j|\sigma(t) = i) = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & i = j, \end{cases}$$

where $\Delta > 0$, $N$ is the cardinality of $\mathcal{P}$, $\gamma_{ij} \geq 0$ for $i \neq j$, and $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. Such switching signals are called *Markovian switching signals*.

**Theorem 3.3.** *If all the conditions in Theorem 3.2 are satisfied, then the consensus protocol (2.6) leads to mean square average-consensus for the agents in (2.2) under any Markovian switching signals.*

Note that both Theorems 3.2 and 3.3 require that each of the digraphs is strongly connected and balanced digraph, while the switching can be arbitrary. This is in accordance with the stability theory for switched systems, where stability under arbitrary switching must imply each of the subsystems itself is stable. Relaxed conditions such as joint connectivity together with some constraints on the switching signals can also lead to consensus under switching topology (see, e.g., Hong, Gao, Cheng, & Hu, 2007; Jadbabaie et al., 2003; Li & Zhang, 2010).

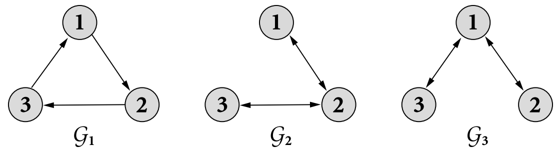


**Fig. 2.** Three different network topologies of 3-agents.

## 4. Simulation results

Consider dynamical networks of three agents. Fig. 2 shows three different topologies denoted by the family $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$. While all digraphs in the figure have 0–1 weights, they are also all strongly connected and balanced. The intensity of the measurement noises satisfies $\sigma_{ij} = 1$ for all $a_{ji} = 1$ $(i, j \in \mathcal{I})$. It can be calculated that $\lambda_2(\hat{\mathcal{L}}_1) = 1.5$, $\lambda_2(\hat{\mathcal{L}}_2) = \lambda_2(\hat{\mathcal{L}}_3) = 1$, and $\|\mathcal{L}_1^2\| = 3$, $\|\mathcal{L}_2^2\| = \|\mathcal{L}_3^2\| = 9$. We simulate two different situations. First, we consider a fixed network topology given by $\mathcal{G}_1$. According to Theorem 3.1, if the communication delays are less than $\lambda_2(\hat{\mathcal{L}}_1)/(\|\mathcal{L}_1^2\|) = 0.5$, then the stochastic consensus protocol (2.6) will lead to mean square average-consensus for the network $\mathcal{G}_1$. The initial states are chosen so that $\text{avg}(x(0)) = 0$. Average-consensus is confirmed by simulation as shown in Fig. 3, where we choose $c(t) = 1/(t+1)$. It is also shown in this figure that if we choose $c(t) \equiv 1$, the noises cannot be attenuated, the states tend to diverge from each other, and average-consensus is not reached. Second, we consider the situation where the network topologies are randomly switching among the three different configurations in $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ according to a continuous-time Markov chain. It follows from Theorem 3.3 that if the delays are less than $\min_{1\le i\le 3} \lambda_2(\hat{\mathcal{L}}_i)/(\max_{1\le i\le 3} \|\mathcal{L}_i\|^2) = 1/9 = 0.1111$, then mean square average-consensus is reached. This is confirmed by simulation as shown in Fig. 4, where we choose $c(t) = 1/(t+1)$. It is also shown in this figure that if we choose $c(t) = 1/(t+1)^2$, while the noises seem to be over attenuated, the states are settled at different values, and again consensus is not reached. Therefore, condition (2.7) on the function $c(t)$ plays a critical role in both attenuating the noise and achieving consensus.

## 5. Proofs

In this section, we present the proofs for the main theorems. Most of the proofs rely on a generalized Gronwall–Bellman–Halanay type inequality, which we will present in the next subsection.

### 5.1. A Gronwall–Bellman–Halanay type inequality

We establish a general Gronwall–Bellman–Halanay type inequality for estimating a function based on a delay differential inequality, which is essential to prove the main theorem and might be of independent interest as well, since it generalizes the classical Halanay inequality in the sense that it can be applied to non-autonomous systems.

**Lemma 5.1.** *Let $t_0$ and $r$ be nonnegative constants. Let $m : [t_0 - r, \infty) \mapsto \mathbb{R}^+$ be continuous and satisfy*

$$D^+m(t) \le \gamma(t) - \mu c(t)m(t) + \lambda c(t) \sup_{-r\le s\le 0} m(t+s), \tag{5.1}$$

*on $[t_0, \infty)$, where $\gamma$ and $c$ are piecewise continuous functions with $\gamma(t) \ge 0$ and $c(t) \in (0, 1]$ for all $t \ge 0$, and $\mu$ and $\lambda$ are constants satisfying $\mu > \lambda > 0$. Then*

$$m(t) \le m_0 \exp\left\{-\rho \int_{t_0}^{t} c(s)ds\right\} + \int_{t_0}^{t} \exp\left\{-\rho \int_{s}^{t} c(r)dr\right\} \gamma(s)ds, \tag{5.2}$$

*holds on $[t_0, \infty)$, where $\rho > 0$ is the root of $-\rho = -\mu + \lambda e^{\rho r}$ and $m_0 = \sup_{-r\le s\le 0} m(t_0 + s)$.*

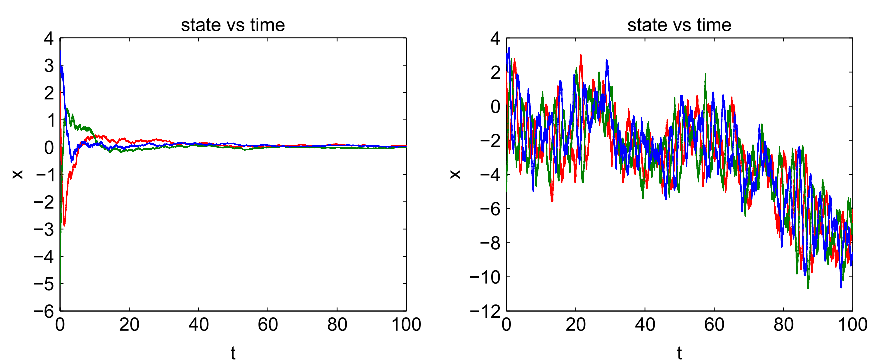


**Fig. 3.** Simulation results for fixed topology $\mathcal{G}_1$: the left figure shows results for $c(t) = 1/(t+1)$ and $\tau = 0.499$; the right figure shows results for $c(t) = 1$ and $\tau = 0.499$.

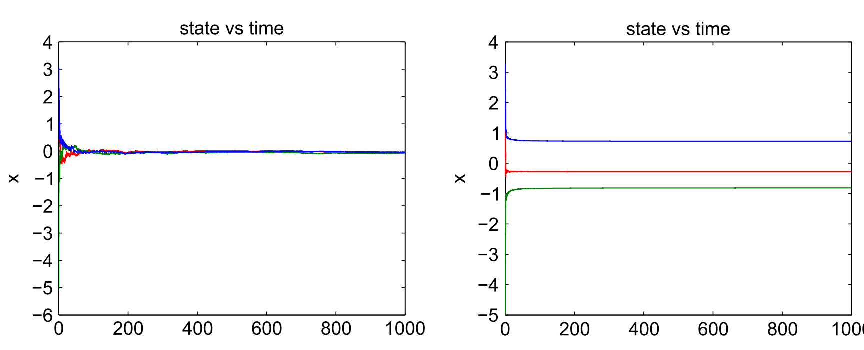


**Fig. 4.** Simulation results for a Markovian switching topology among $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ driven by the generator $\Gamma = [-0.5\ 0.2\ 0.3; 0.4 - 0.7\ 0.3; 0.2\ 0.55 - 0.75]$: the left figure shows results for $c(t) = 1/(t+1)$ and $\tau = 0.111$; the right figure shows results for $c(t) = 1/(t+1)^2$ and $\tau = 0.111$.

**Proof.** Define

$$u(t)=\begin{cases}\text{RHS of (5.2)}, & t\in[t_0,\infty),\\ m_0, & t\in[t_0-r,t_0),\end{cases}$$

and

$$t^*=t^*(\varepsilon)=\inf\{t\in[t_0-r,\infty):\ m(t)>u(t)+\varepsilon\},$$

where $\varepsilon>0$ is an arbitrary positive constant. Note that, for $t\in[t_0-r,t_0]$, we have

$$m(t)\le\sup_{-r\le s\le 0}m(t_0+s)=m_0=u(t).$$

Hence $t^*\ge t_0$. If $t^*=\infty$ for all $\varepsilon$, then the lemma is proved. Suppose $t^*\in[t_0,\infty)$ for some $\varepsilon>0$. By the definition of $t^*$ and the continuity of $m(t)$, we have $m(t^*)=u(t^*)+\varepsilon$ and

$$\begin{aligned}\sup_{-r\le s\le 0}m(t^*+s)&\le\sup_{-r\le s\le 0}u(t^*+s)+\varepsilon\\ &\le e^{\rho r}u(t^*)+\varepsilon<e^{\rho r}m(t^*).\end{aligned}\tag{5.3}$$

Therefore, by (5.1) and (5.3),

$$\begin{aligned}D^+[m-u](t^*)&<(-\mu+\lambda e^{\rho r}+\rho)c(t^*)m(t^*)-\rho c(t^*)\varepsilon\\ &=-\rho c(t^*)\varepsilon<0,\end{aligned}\tag{5.4}$$

which contradicts how $t^*$ is defined. Therefore, we must have $t^*=\infty$ for all $\varepsilon>0$ and the lemma is proved. □

### 5.2. Proof of Theorem 3.1

Applying Itô's formula (see, e.g., Revuz & Yor, 1999, Theorem 3.3, Chatper IV) to $V(t)$ in view of (3.2), we have

$$\begin{aligned}dV(t)&=-2c(t)\delta^T(t)\mathcal{L}\delta(t)dt\\ &\quad+2c(t)\delta^T(t)\mathcal{L}[\delta(t)-\delta(t-\tau(t))]dt\\ &\quad+C_0c^2(t)dt+2c(t)\delta^T(t)(I-J)\Theta dW(t),\end{aligned}\tag{5.5}$$

where

$$C_0=\mathrm{trace}[(I-J)^2\Theta\Theta^T].\tag{5.6}$$

Eq. (5.5) implies that

$$\begin{aligned}E(V(t))-E(V(\tau))&=-2\int_\tau^t c(s)E[\delta^T(s)\mathcal{L}\delta(s)]ds\\ &\quad+2\int_\tau^t c(s)E\{\delta^T(s)\mathcal{L}[\delta(s)-\delta(s-\tau(s))]\}ds\\ &\quad+C_0\int_\tau^t c^2(s)ds.\end{aligned}\tag{5.7}$$

Let $m(t)=E(V(t))$ for $t\ge 0$. Writing the above integral equation in differential form gives

$$\begin{aligned}D^+m(t)&=-2c(t)E[\delta^T(t)\mathcal{L}\delta(t)]+C_0c^2(t)\\ &\quad+2c(t)E\{\delta^T(t)\mathcal{L}[\delta(t)-\delta(t-\tau(t))]\}.\end{aligned}\tag{5.8}$$

Note that

$$E[\delta^T(t)\mathcal{L}\delta(t)]\ge\lambda_2(\hat{\mathcal{L}})E(V(t)),$$

and

$$\begin{aligned}&2E\{\delta^T(t)\mathcal{L}[\delta(t)-\delta(t-\tau(t))]\}\\ &\quad\le\varepsilon E(V(t))+\frac{1}{\varepsilon}E\{\mathcal{L}[\delta(t)-\delta(t-\tau(t))]\}^2,\end{aligned}$$

where $\varepsilon>0$ is a constant to be determined later. On the other hand, Eq. (3.4) implies that

$$\begin{aligned}&E\{\mathcal{L}[\delta(t)-\delta(t-\tau(t))]\}^2\\ &\le(1+\beta)E\left|\int_{t-\tau(t)}^t c(s)\mathcal{L}^2\delta(s-\tau(s))ds\right|^2\\ &\quad+\left(1+\frac{1}{\beta}\right)E\left|\int_{t-\tau(t)}^t c(s)\mathcal{L}(I-J)\Theta dW(s)\right|^2\\ &\le(1+\beta)\tau\|\mathcal{L}^2\|^2\int_{t-\tau}^t E|\delta(s-\tau(s))|^2ds\\ &\quad+\left(1+\frac{1}{\beta}\right)\tau C_0\|\mathcal{L}^2\|\int_{t-\tau}^t c^2(s)ds\\ &\le(1+\beta)\tau^2\|\mathcal{L}^2\|^2\sup_{-2r\le s\le 0}E(V(t+s))\\ &\quad+\left(1+\frac{1}{\beta}\right)\tau C_0\|\mathcal{L}^2\|\int_{t-\tau}^t c^2(s)ds,\end{aligned}$$

where $\beta>0$ is a constant to be chosen later. Putting the above three estimates together into (5.8) and setting $\varepsilon=\tau\|\mathcal{L}^2\|$, we obtain

$$\begin{aligned}D^+m(t)&\le-2\lambda_2(\hat{\mathcal{L}})c(t)m(t)\\ &\quad+2(1+\beta)\tau\|\mathcal{L}^2\|c(t)\sup_{-2r\le s\le 0}m(t+s)\\ &\quad+\left(1+\frac{1}{\beta}\right)C_0^2\int_{t-\tau}^t c^2(r)dr+C_0c^2(t),\quad t\ge\tau.\end{aligned}$$

Inequality (3.1) implies that we can choose $\beta>0$ sufficiently small such that $2\lambda_2(\hat{\mathcal{L}})>(2+\beta)\tau\|\mathcal{L}^2\|$. Therefore, there exists $\rho>0$ such that $-2\lambda_2(\hat{\mathcal{L}})+(2+\beta)\tau\|\mathcal{L}^2\|e^{2\rho\tau}=-\rho$. Lemma 5.1 shows that

$$\begin{aligned}m(t)&\le m_0\exp\left\{-\rho\int_\tau^t c(s)ds\right\}\\ &\quad+\int_\tau^t\exp\left\{-\rho\int_s^t c(r)dr\right\}\gamma(s)ds,\end{aligned}\tag{5.9}$$

on $[\tau,\infty)$ with

$$m_0=\sup_{-\tau\le s\le\tau}m(s),$$

$$\gamma(t)=\left(1+\frac{1}{\beta}\right)C_0^2\int_{t-\tau}^t c^2(r)dr+C_0c^2(t).$$

It follows from (2.7) that

$$\exp\left\{-\rho\int_\tau^t c(s)ds\right\}\to 0,\tag{5.10}$$

as $t\to\infty$. On the other hand, note that, for all $t\ge T\ge\tau$, we have

$$\begin{aligned}&\int_\tau^t\exp\left\{-\rho\int_s^t c(r)dr\right\}\int_{s-\tau}^s c^2(r)drds\\ &=\int_\tau^T\exp\left\{-\rho\int_s^t c(r)dr\right\}\int_{s-\tau}^s c^2(r)drds\end{aligned}$$

$$+\int_T^t \exp\left\{-\rho\int_s^t c(r)dr\right\}\int_{s-\tau}^s c^2(r)drds$$

$$\le \exp\left\{-\rho\int_T^t c(r)dr\right\}\int_\tau^\infty\int_{s-\tau}^s c^2(r)drds$$

$$+\int_T^\infty\int_{s-\tau}^s c^2(r)drds, \tag{5.11}$$

and, similarly,

$$\int_\tau^t \exp\left\{-\rho\int_s^t c(r)dr\right\}c^2(s)ds$$

$$\le \exp\left\{-\rho\int_T^t c(r)dr\right\}\int_\tau^\infty c^2(s)ds+\int_T^\infty c^2(s)ds. \tag{5.12}$$

In view of (2.7), for fixed $T$,

$$\exp\left\{-\rho\int_T^t c(s)ds\right\}\to 0, \tag{5.13}$$

as $t\to\infty$, and

$$\int_\tau^\infty\int_{s-\tau}^s c^2(r)drds \le \int_0^\infty c^2(r)\int_r^{r+\tau} dsdr$$

$$=\tau\int_0^\infty c^2(r)dr<\infty. \tag{5.14}$$

Therefore, both

$$\int_T^\infty\int_{s-\tau}^s c^2(r)drds\to 0,\qquad \int_T^\infty c^2(s)ds\to 0, \tag{5.15}$$

as $T\to\infty$. Putting (5.13)–(5.15) into (5.11) and (5.12), we have shown

$$\int_\tau^t \exp\left\{-\rho\int_s^t c(r)dr\right\}\gamma(s)ds\to 0, \tag{5.16}$$

as $t\to\infty$. Combining (5.10) and (5.16) into (5.9), we finally get $E(V(t))=m(t)\to 0$ as $t\to\infty$.

Now note that

$$\alpha(t)=\frac{1}{n}\mathbf{1}^Tx(t)=\frac{1}{n}\mathbf{1}^Tx(0)+\int_0^t\frac{1}{n}c(s)\mathbf{1}^T(I-J)\Theta dW(s).$$

Since

$$E\left\{\int_0^t c(s)\frac{1}{n}\mathbf{1}^T(I-J)\Theta dW(s)\right\}^2=\frac{1}{n}C_0\int_0^t c^2(s)ds$$

$$<\int_0^\infty c^2(s)ds<\infty,$$

where $C_0$ is defined by (5.6), it follows that $\alpha$ is an $L^2$-bounded martingale. Doob's martingale convergence theorem (see, e.g., Revuz & Yor, 1999, Theorem 2.10, Chapter II) guarantees there exists a random variable $x^*$ with $E|x^*|^2<\infty$ such that

$$\lim_{t\to\infty}\alpha(t)=x^* \text{ a.s.}\quad\text{and}\quad \lim_{t\to\infty}E|\alpha(t)-x^*|^2=0.$$

Moreover, $E(x^*)=E(\alpha(t))=E(\alpha(0))=\sum_{i=1}^n x_i(0)/n$. On the other hand, for all $i\in\mathcal{I}$,

$$|x_i(t)-x^*|^2\le 2|x_i(t)-\alpha(t)|^2+2|\alpha(t)-x^*|^2$$

$$=2|\delta_i(t)|^2+2|\alpha(t)-x^*|^2$$

$$\le 2V(t)+2|\alpha(t)-x^*|^2.$$

It follows from both $E(V(t))\to 0$ and $E|\alpha(t)-x^*|^2\to 0$ that

$$E|x_i(t)-x^*|^2\to 0,$$

as $t\to\infty$. Moreover, it is easy to check that

$$E|x_i(t)|^2\le 2EV((t))+2E\alpha^2(t)<\infty,\quad t\ge 0.$$

Therefore, mean square consensus is reached and the proof is complete. □

### 5.3. Proof of Theorem 3.2

Let $\sigma(t)$, $t\ge 0$, be a given switching signal. Then $V(t)$ can serve as a common Lyapunov function for the displacement dynamics

$$d\delta(t)=c(t)[-\mathcal{L}_{\sigma(t)}\delta(t-\tau(t))dt+(I-J)\Theta dW(t)], \tag{5.17}$$

which follows from (3.6). Repeating the same argument as in the proof of Theorem 3.1, we can obtain

$$D^+m(t)\le -2\min_{p\in\mathcal{P}}\lambda_2(\hat{\mathcal{L}}_p)c(t)m(t)$$

$$+(2+\beta)\tau\max_{p\in\mathcal{P}}\|\mathcal{L}_p\|^2c(t)\sup_{-2r\le s\le 0}m(t+s)$$

$$+\left(1+\frac{1}{\beta}\right)C_0^2\int_{t-\tau}^t c^2(r)dr+C_0c^2(t),\quad t\ge\tau.$$

Since we can choose $\beta>0$ such that $2\min_{p\in\mathcal{P}}\lambda_2(\hat{\mathcal{L}}_p)>(2+\beta)\tau\max_{p\in\mathcal{P}}\|\mathcal{L}\|^2$, there exists $\rho>0$ such that $-2\lambda_2(\hat{\mathcal{L}})+(2+\beta)\tau\|\mathcal{L}\|^2e^{2\rho\tau}=-\rho$. Lemma 5.1 implies that the same estimate (5.9) holds and the rest of the proof is essentially the same as the proof of Theorem 3.1. □

### 5.4. Proof of Theorem 3.3

The proof is essentially the same as that of Theorem 3.2, except that we should apply the generalized Itô's formula Mao and Yuan (2006, Theorem 1.45) due to the Markovian switching. Since a common Lyapunov function $V=|\delta|^2$ is used, we obtain the same integral equation as (5.7) for the expectation $E(V(t))$ with $\mathcal{L}_{\sigma(s)}$ in place of $\mathcal{L}$. The rest of the proof is the same. □

## 6. Conclusions

We have investigated the average-consensus problem of networked multi-agents systems subject to measurement noises. A time-varying consensus protocol that takes into the account both the measurement noises and general time-varying communication delays has been proposed. We have considered general networks with fixed topology, with arbitrary deterministic switching topology, and with Markovian switching topology. For each of these three cases, we have obtained sufficient conditions under which the proposed consensus protocol leads to mean square average-consensus. The sufficient conditions provide explicit delay upper bounds guaranteeing mean square average-consensus in terms of the graph Laplacians. We conclude our paper by pointing out some possible future research.

First, this paper focuses only on the mean square average-consensus. For future research, it would be interesting to propose consensus protocols to reach both general $p$th moment consensus and almost sure consensus.

Moreover, while characterizing exact delay bounds for stability of general linear time-delay systems remain challenging issues, obtaining exact delay upper bounds for consensus are of practical importance. In Olfati-Saber and Murray (2004), the exact delay bound for average-consensus is obtained for a fixed digraph network with a single constant delay, and in Bliman and Ferrari-Trecate (2008), exact delay bounds for average consensus are obtained for a fixed undirected network with a single time-varying delay or multiple constant delays. It would be interesting to know if these bounds are still optimal under the measurement noises considered in the current paper. If so, one might be able to obtain necessary and sufficient conditions for mean square consensus despite both measurement noises and communication delays following the treatment in Li and Zhang (2009) for networked systems without communication time-delays. It would also be of practical importance to deal with more general (nonuniform, multiple, time-varying or distributed) delays and communication noises occurring at the same time.

Finally, when considering switching topology network, it is possible that some modes of the network fail to have a strongly connected and/or balanced network. These modes may be characterized as unstable modes and it would be interesting to investigate if the ideas from stability analysis of switched systems with both stable and unstable subsystems (see, e.g. Liu, Liu, & Xie, 2009) can be borrowed here to investigate consensus problems under such situations.

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## References


Bauso, D., Giarre, L., & Pesenti, R. (2009). Consensus for networks with unknown but bounded disturbances. *SIAM Journal on Control and Optimization*, *48*, 1756–1770.

Bliman, P. A., & Ferrari-Trecate, G. (2008). Average consensus problems in networks of agents with delayed communications. *Automatica*, *44*, 1985–1995.

Borkar, V., & Varaiya, P. (1982). Asymptotic agreement in distributed estimation. *IEEE Transactions on Automatic Control*, *27*, 650–655.

Cao, M., Morse, A. S., & Anderson, B. D. O. (2008a). Reaching a consensus in a dynamically changing environment: a graphical approach. *SIAM Journal on Control and Optimization*, *47*, 575–600.

Cao, M., Morse, A. S., & Anderson, B. D. O. (2008b). Reaching a consensus in a dynamically changing environment: convergence rates, measurement delays, and asynchronous events. *SIAM Journal on Control and Optimization*, *47*, 601–623.

Carli, R., Fagnani, F., Speranzon, A., & Zampieri, S. (2006). Communication constraints in coordinated consensus problems. In *Proceedings of the American control conference. Minneapolis, MN* (pp. 4189–4194).

de Castro, G. A., & Paganini, F. (2004). Convex synthesis of controllers for consensus. In *Proceedings of the American control conference. Boston, MA* (pp. 4933–4938).

DeGroot, M. H. (1974). Reaching a consensus. *Journal of the American Statistical Association*, *69*, 118–121.

Fax, J. A., & Murray, R. M. (2004). Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, *49*, 1465–1476.

Fiedler, M. (1973). Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, *23*(98), 298–305.

Hatano, Y., & Mesbahi, M. (2005). Agreement over random networks. *IEEE Transactions on Automatic Control*, *50*, 1867–1872.

Hong, Y., Gao, L., Cheng, D., & Hu, J. (2007). Lyapunov-based approach to multiagent systems with switching jointly connected interconnection. *IEEE Transactions on Automatic Control*, *52*, 943–948.

Huang, M., & Manton, J. H. (2009). Coordination and consensus of networked agents with noisy measurements: stochastic algorithms and asymptotic behavior. *SIAM Journal on Control and Optimization*, *48*, 134–161.

Huang, M., & Manton, J. H. (2010a). Stochastic consensus seeking with noisy and directed inter-agent communication: fixed and randomly varying topologies. *IEEE Transactions on Automatic Control*, *55*, 235–241.

Huang, M., & Manton, J. H. (2010b). Stochastic consensus over noisy networks with Markovian and arbitrary switches. *Automatica*, *46*, 1571–1583.

Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, *48*, 988–1001.

Li, T., & Zhang, J. F. (2009). Mean square average-consensus under measurement noises and fixed topologies: necessary and sufficient conditions. *Automatica*, *45*, 1929–1936.

Li, T., & Zhang, J. F. (2010). Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises. *IEEE Transactions on Automatic Control*, *55*, 2043–2057.

Lin, P., & Jia, Y. (2009). Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatica*, *45*, 2154–2158.

Liu, J., Liu, X., & Xie, W.-C. (2009). Uniform stability of switched nonlinear systems. *Nonlinear Analysis: Hybrid Systems*, *3*, 441–454.

Lu, J., Ho, Daniel W. C., & Kurths, Jürgen (2009). Consensus over directed static networks with arbitrary finite communication delays. *Physical Review E*, *80*, 066121.

Lynch, N. A. (1997). *Distributed algorithms*. San Francisco, CA: Morgan Kaufmann.

Mao, X., & Yuan, C. (2006). *Stochastic differential equations with Markovian switching*. Imperial College Press.

Moreau, L. (2005). Stability of multi-agent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, *50*, 169–182.

Munz, U., Papachristodoulou, A., & Allgower, F. (2010). Delay robustness in consensus problems. *Automatica*, *46*, 1252–1265.

Olfati-Saber, R., Fax, J. A., & Murray, R. M. (2007). Consensus and cooperation in networked multi-agent system. *Proceedings of the IEEE*, *95*, 215–233.

Olfati-Saber, R., & Murray, R. M. (2004). Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, *49*, 1520–1533.

Ren, W., & Beard, R. W. (2005). Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, *50*, 655–661.

Ren, W., Beard, R. W., & Kingston, D. B. (2005). Multi-agent Kalman consensus with relative uncertainty. In *Proceedings of the American control conference. Portland, OR* (pp. 1865–1870).

Revuz, D., & Yor, M. (1999). *Continuous martingales and Brownian motion*. Springer Verlag.

Tian, Y., & Liu, C. (2008). Consensus of multi-agent systems with diverse input and communication delays. *IEEE Transactions on Automatic Control*, *53*, 2122–2128.

Tsitsiklis, J. N., & Athans, M. (1984). Convergence and asymptotic agreement in distributed decision problems. *IEEE Transactions on Automatic Control*, *29*, 42–50.

Tsitsiklis, J. N., Bertsekas, D. P., & Athans, M. (1986). Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, *31*, 803–812.

Wang, W., & Slotine, J. (2006). Contraction analysis of time-delayed communications and group cooperation. *IEEE Transactions on Automatic Control*, *51*, 712–717.

Xiao, L., Boyd, S., & Kim, S.-J. (2007). Distributed average consensus with least-mean-square deviation. *Journal of Parallel and Distributed Computing*, *67*, 33–46.

Xiao, F., & Wang, L. (2008). Consensus protocols for discrete-time multi-agent systems with time-varying delays. *Automatica*, *44*, 2577–2582.

Yang, B., & Fang, H. (2010). Forced consensus in networks of double integrator systems with delayed input. *Automatica*, *46*, 629–632.

Zhu, W., & Cheng, D. (2010). Leader-following consensus of second-order agents with multiple time-varying delays. *Automatica*, *46*, 1994–1999.


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