

## A NEW UNIFIED CHAOTIC SYSTEM AND ITS IMPULSIVE SYNCHRONIZATION

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**Abstract.** This paper introduces a new unified chaotic system that connects the Chen's chaotic system to the classical Lorenz chaotic system through the Lü chaotic system. The continued transition from the Lorenz and the Chen systems is controlled by  $|x|$ , thereby the new attractor is composed by two parts, one part is half a Lorenz attractor, another part is half a Chen attractor. The dynamical behavior of the new unified system is investigated. An electronic circuit is also designed and built to confirm its dynamics of the new chaotic system. Then impulsive synchronization of the new unified chaotic system is studied.

**Keywords.** chaos, unified chaotic system, chaotic circuit, impulsive synchronization.

### 1 Introduction

Chaotic attractors are found to be very useful for various applications, such as network modeling [4], encryption and secure communication [3,5-15] and so on. This provides a strong motivation for the current research on exploiting some new chaotic attractors and their implementations. In 1999, a new chaotic attractor, known as Chen's attractor, is found [2]. It is proved to be topologically different from the classical Lorenz chaotic system [1,2,17]. Both systems can be expressed in terms of a generalized Lorenz system [1], consisting of linear and nonlinear parts, where the linear part is a constant matrix with  $A = [a_{ij}]_{3 \times 3}$ . According to [1], the Lorenz system and Chen system satisfy the conditions  $a_{12}a_{21} > 0$  and  $a_{12}a_{21} < 0$ , respectively, and in this sense they are dual systems. Recently, the transition between Lorenz and Chen is bridged with the Lü system, under the condition of  $a_{12}a_{21} = 0$  [18]. In a canonical form, all these topologically non-equivalent 3-dimensional autonomous quadratic chaotic systems of various structures can be classified as a family of generalized hyperbolic chaotic systems [1]. Recently, a unified chaotic system was proposed in [19] for the generation of all these three classes of attractors, which has a broad spectrum of chaotic behaviors with

the Lorenz and the Chen systems as two extremes of the spectrum of the key system parameter.

On the other hand, chaos synchronization has also become an active area of research due to its potential applications to secure communication [5-15, 20-22, 24-31]. Several chaos-based secure communication schemes have been proposed. In these schemes, message signals are masked or modulated (encrypted) by using chaotic signals and the resulting encrypted signals are transmitted across a public channel. Perfect synchronization is usually expected to recover the information signals. Recently, another synchronization technique, based on impulsive control, has been reported and developed in [8-12, 28-30]. The technique allows the coupling and synchronization of two or more chaotic systems by using only small synchronizing impulses. These impulses are samples of the state variables (or functions of the state variables) of the drive system at discrete moments. They drive the response system discretely at these moments.

This paper introduces a new unified chaotic system that connects the Chen's chaotic system to the classical Lorenz chaotic system through the Lü chaotic system. The continued transition from the Lorenz and the Chen systems is controlled by  $|x|$ , thereby the new attractor is composed by two parts, one part is half a Lorenz attractor, another part is half a Chen attractor. The dynamical behavior of the new unified system and its impulsive synchronization are investigated respectively. In this paper, an electronic circuit is also designed and built to confirm its dynamics of the new chaotic system.

The rest of the paper is organized as follows. In the next section, the mathematical model of the unified chaotic system is introduced. The dynamical behavior of the new unified system is investigated in Section 3. An electronic circuit realization for the system and some experimental observations are then presented in Section 4. In Section 5, impulsive synchronization of the new unified chaotic system is investigated. Finally, concluding remarks are given in Section 6.

## 2 Mathematical Model of the New Unified Chaotic System

The new unified chaotic system is described as follows:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = \gamma|x| - xz + \beta y \\ \dot{z} = xy - bz \end{cases} \quad (1)$$

where  $(x, y, z) \in \mathcal{R}^3$ ,  $a, b, \beta, \gamma \in \mathcal{R}^1 > 0$ .

According to the canonical-form criterion formulated in [1], one has:

(i) when  $x > 0$ , system (1) can be reformulated form

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = \gamma x - xz + \beta y \\ \dot{z} = xy - bz \end{cases} \quad (2)$$

or in the following canonical form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -a & a & 0 \\ \gamma & \beta & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (3)$$

satisfying condition  $a_{12}a_{21} = a \times \gamma > 0$ , so it is a generalized Lorenz system.

(ii) when  $x < 0$ , system (1) can be reformulated form

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -\gamma x - xz + \beta y \\ \dot{z} = xy - bz \end{cases} \quad (4)$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -a & a & 0 \\ -\gamma & \beta & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (5)$$

satisfying condition  $a_{12}a_{21} = a \times \gamma < 0$ , so it is a generalized Chen system.

It can be seen that if the sign of the term  $a_{12}a_{21}$  in system (1) can be controlled to change alternatively between "+" and "-" by  $|x|$ , then the system dynamics will change (in some sense, switch) among different chaotic attractors.

Choosing  $a = 35$ ,  $\gamma = 5$ ,  $\beta = 27.5$  and  $b = 3$ , the corresponding chaotic attractors are depicted in Figs. 1 (a)-(d).

### 3 Dynamical Behaviors of the Unified System

#### 3.1 Some Basic Properties

The new hyperchaotic attractor has some different qualitative properties from both the Lorenz and Chen systems along with sharing several important qualitative properties. They are listed as follows.

##### 3.1.1 Symmetry and Invariance

First, It is easy to see that system (1) does not have a property of symmetry. The equation is variant under the following transformation:

$$(x, y, z) \mapsto (-x, -y, z).$$

Next, it is clear that the z-axis itself is an orbit, i.e. if  $x = y = 0$  at  $t = 0$  then  $x = y = 0$  for all  $t > 0$ . Moreover, the orbit on the z-axis tends to the origin as  $t \rightarrow \infty$ .

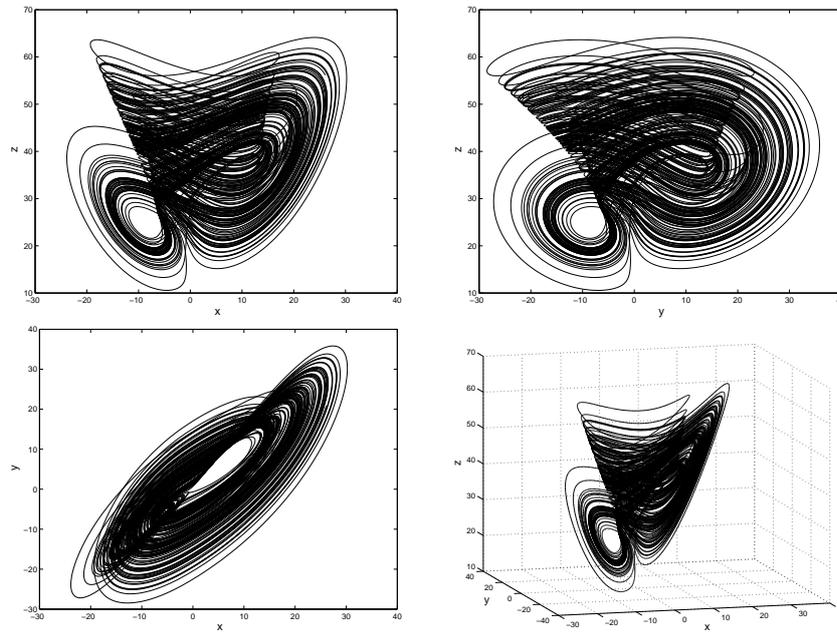


Figure 1: Phase portraits of system (1) with  $a = 35$ ,  $\gamma = 5$ ,  $\beta = 27.5$  and  $b = 3$ : (a)  $x - z$  plane, (b)  $y - z$  plane, (c)  $x - y$  plane, (d)  $x - y - z$  plane.

### 3.1.2 Dissipativity and the Existence of Attractor

Let us consider a volume in a certain domain  $D_0$  of the state space. Notice that

$$\nabla V = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -a + \beta - b.$$

We always use a set of parameters satisfying  $-a + \beta - b < 0$ , such as  $(a, \beta, b) = (35, 27.5, 3)$ , so that the dynamical system (1) is guaranteed to be dissipative, with an exponential contraction rate

$$\frac{dV}{dt} = (-a + \beta - b)V.$$

That is, a volume element  $V_0$  is contracted by the flow into a volume element  $V_0 e^{(-a+\beta-b)t}$  in time  $t$ . This means that each volume containing the system trajectory shrinks to zero as  $t \rightarrow \infty$  at an exponential rate  $-a + \beta - b$ . Therefore, all system orbits are ultimately confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor.

### 3.2 Equilibria and Stability

To obtain the equilibria, we set the vector field of system (1) to zero, which leads to

$$\begin{aligned} a(y - x) &= 0 \\ \gamma|x| - xz + \beta y &= 0 \\ xy - bz &= 0. \end{aligned}$$

When  $\beta > \gamma > 0$ , system (1) has three equilibria:

$$\begin{aligned} &S_0(0, 0, 0); \\ x > 0 &: S_1 \left( \sqrt{b(\beta + \gamma)}, \sqrt{b(\beta + \gamma)}, (\beta + \gamma) \right), \\ x < 0 &: S_2 \left( -\sqrt{b(\beta - \gamma)}, -\sqrt{b(\beta - \gamma)}, (\beta - \gamma) \right). \end{aligned}$$

For the equilibria,  $S_1$  and  $S_2$  are symmetrically placed with respect to the  $z$ -axis.

Let us now study their stability properties, i.e. the nature of the eigenspaces presented in the neighborhood of each equilibrium point. The Jacobian matrix of system (1) can be written as

$$\begin{bmatrix} -a & a & 0 \\ \gamma \text{sign}(x) - z & \beta & -x \\ y & x & -b \end{bmatrix} \quad x \neq 0. \tag{6}$$

For simplicity, let  $a = 35$ ,  $\beta = 27.5$  and  $b = 3$ , and let  $\gamma$  vary.

(i) Linearizing system (1) at the equilibrium point  $S_1$ , yields the following characteristic equation:

$$f(\lambda) = \lambda^3 + (a + b - \beta)\lambda^2 + b(a + \gamma)\lambda + 2ab(\beta + \gamma) = 0. \tag{7}$$

Let

$$\begin{aligned} A &= a + b - \beta = 10.5 > 0, \\ B &= b(a + \gamma) = 3 \times (35 + \gamma) > 0, \\ C &= 2ab(\beta + \gamma) = 210 \times (3 + \gamma) > 0. \end{aligned}$$

From Eq. 7, one may assume that  $\lambda = -\frac{A}{3} + \Lambda$ . This yields

$$f(\Lambda) = \Lambda^3 + P\Lambda + Q$$

where  $P = -\frac{A^2}{3} + B$  and  $Q = \frac{2A^3}{27} - \frac{AB}{3} + C$ . This third-order polynomial in  $\Lambda$  can be solved by using the Cardan formula, where one may set

$$\Delta = 4P^3 + 27Q^2.$$

Obviously, when  $\gamma > 0$ ,

$$\begin{aligned} P &= -\frac{A^2}{3} + B \\ &= -\frac{1}{3}(a + b - \beta)^2 + b(a + \gamma) \\ &= 3\gamma + 68.25 \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} Q &= \frac{2A^3}{27} - \frac{AB}{3} + C \\ &= \frac{2}{27}(a + b - \beta)^3 - \frac{b}{3}(a + b - \beta)(a + \gamma) + 2ab(\beta + \gamma) \\ &= 5493.3 + 199.5\gamma \\ &> 0. \end{aligned}$$

So that

$$\Delta = 4P^3 + 27Q^2 > 0.$$

Consequently, Eq. (7) has a unique real eigenvalue:

$$\begin{aligned} \lambda_R &= -\frac{A}{3} + \Lambda_R \\ &= -\frac{A}{3} + \left(-\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}\right)^{1/3} + \left(-\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}\right)^{1/3} \\ &= -\frac{A}{3} + \left(-\frac{Q}{2} + \sqrt{\frac{\Delta}{4 \times 27}}\right)^{1/3} + \left(-\frac{Q}{2} - \sqrt{\frac{\Delta}{4 \times 27}}\right)^{1/3} \end{aligned} \quad (8)$$

along with two complex complex conjugate eigenvalues:

$$\begin{aligned} (\lambda_C)_\pm &= \sigma \pm \psi i \\ &= -\frac{A}{3} + (\Lambda_C)_\pm \\ &= -\frac{A}{3} - \frac{\Lambda_R}{2} \pm \frac{i}{2} \sqrt{4P + 3(\Lambda_R)^2}. \end{aligned} \quad (9)$$

Since  $A > 0$ ,  $Q > 0$  and  $\Delta > 0$ , we obtain  $\lambda_R < 0$ ,  $\sigma = 0$ .

Since  $\lambda_R + 2\sigma = -A$ , and  $\lambda_R < 0$ ,  $\sigma > 0$ , we have  $|\lambda_R| - 2\sigma = A > 0$ . The characteristic eigenvalues of the equilibrium point satisfy the Shil'nikov inequalities [11], that is,

$$\lambda_R \sigma < 0 \quad \text{and} \quad |\lambda_R| > \sigma > 0.$$

Therefore, the equilibrium point is a saddle-foci with an instability index of 2, and the negative real eigenvalue is associated with a one-dimensional stable manifold, whereas both complex conjugate eigenvalues, with a two-dimensional unstable manifold in which trajectories are spiraling outwards.

(ii) Linearizing system (1) at the other equilibrium point  $S_2$ , yields the following characteristic equation:

$$f(\lambda) = \lambda^3 + (a + b - \beta)\lambda^2 + (b(a + \gamma) + 2a\gamma)\lambda + 2ab(\beta + 2\gamma) = 0. \quad (10)$$

Let

$$\begin{aligned} A &= a + b - \beta = 10.5 > 0, \\ B &= b(a + \gamma) + 2a\gamma > 0, \\ C &= 2ab(\beta + 2\gamma) > 0. \end{aligned}$$

and

$$\begin{aligned} P &= -\frac{A^2}{3} + B \\ &= -\frac{1}{3}(a + b - \beta)^2 + b(a + \gamma) + 2a\gamma \\ &= 68.25 + 73\gamma \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} Q &= \frac{2A^3}{27} - \frac{AB}{3} + C \\ &= \frac{2}{27}(a + b - \beta)^3 - \frac{1}{3}(a + b - \beta)(b(a + \gamma) + 2a\gamma) + 2ab(\beta + 2\gamma) \\ &= 164.5\gamma + 5824 \\ &> 0. \end{aligned}$$

Obviously,

$$\Delta = 4P^3 + 27Q^2 > 0.$$

Similarly, we obtain the characteristic eigenvalues of the equilibrium point  $S_2$ , satisfy the Shil'nikov inequalities [11], that is,  $\lambda_R\sigma < 0$  and  $|\lambda_R| > \sigma > 0$ .

Therefore, the equilibrium point is a saddle-foci with an instability index of 2, and the negative real eigenvalue is associated with a one-dimensional stable manifold, whereas both complex conjugate eigenvalues, with a two-dimensional unstable manifold in which trajectories are spiraling outwards.

## 4 Circuit Design for the New Unified Chaotic System

The new unified chaotic system (1) has also been confirmed by an electronic circuit, as shown in Fig. 2. The operational amplifiers and associated circuitry perform the basic operations of addition, subtraction, and integration. The nonlinear terms in the equation are implemented with the analog multipliers AD633.

The occurrence of the new unified attractor can be clearly seen from Figs. 3 (a)-(c). By comparing it with Figs. 3(a)-(c), it can be concluded that a good qualitative agreement between the numerical simulation and the experimental realization is obtained.

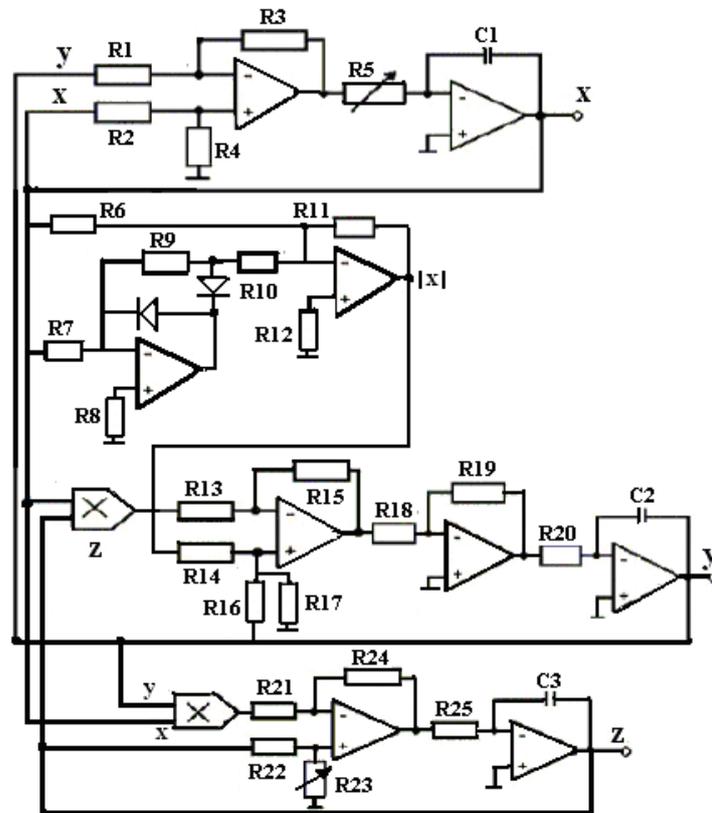


Figure 2: Circuitry realization of the new unified system (1) Resistors (k):  $R_1, R_2, R_4, R_6, R_7, R_9, R_{10}, R_{11}, R_{15}, R_{18}, R_{19}, R_{20}, R_{22}, R_{24}, R_{25}=10$ ;  $R_5=3.69$ ;  $R_{13}, R_{14}, R_{21}=1$ ;  $R_8=5$ ;  $R_{12}=3.75$ ;  $R_{18}=0.472$  Capacitors (nF):  $C_1, C_2, C_3 = 1$ ; Op-Amps: TL084; Multipliers: AD633.

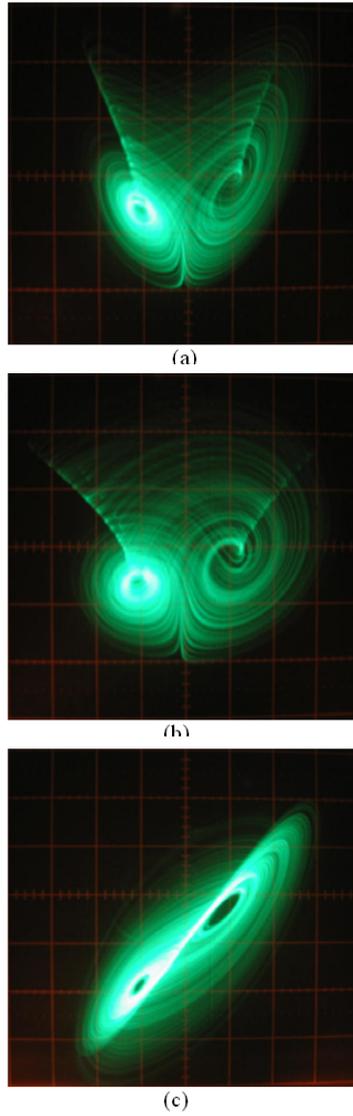


Figure 3: Phase portraits of the unified chaotic system (1) measured from the experimental setup in: (a) x - z plane, (b) y - z plane, (c) x - y plane, the scales of vertical and horizontal axes are 0.5V/div and 0.5 V/div respectively.

## 5 Impulsive Synchronization

In this section, we shall investigate the problem of impulsive synchronization of the new unified chaotic system. Consider the drive system

$$\dot{p} = f(t, p) \quad (11)$$

where  $p = (x, y, z)^T$ ,

$$f(t, p) = \begin{bmatrix} a(y - x) \\ \gamma|x| - xz + \beta y \\ xy - bz \end{bmatrix},$$

$a = 35$ ,  $\gamma = 5$ ,  $\beta = 27.5$  and  $b = 3$ .

The response system is given by

$$\begin{cases} \dot{q} &= f(t, q), & t \neq t_k \\ \Delta q &= -B_k(p - q), & t = t_k, \quad k \in N \end{cases} \quad (12)$$

where  $q = (u, v, w)^T$ ,  $t_1 < t_2 < \dots$ , and  $B_k$  is the impulsive control matrix,  $k = 1, 2, \dots$

Let  $e = p - q$ . Then we say that the system (11) and the system (12) are synchronized by impulsive driving if the trivial solution of the error system

$$\begin{cases} \dot{e} &= f(t, p) - f(t, q), & t \neq t_k \\ \Delta e &= B_k e, & t = t_k, \quad k \in N \end{cases} \quad (13)$$

is attractive.

Since solutions of the chaotic systems (11) and (12) are bounded, there must exist  $\rho_0 > 0$  such that  $\|e\| < \rho_0$ .

Consider the error system (13). Let  $\rho = \rho_0 + 1$ , and choose  $B_k = -0.9I$ ,  $K = 1, 2, \dots$ , then  $\|e + \Delta e\| = \|0.1e\| < \rho_0 < \rho$ . Apparently  $e + \Delta e \in S(\rho)$ .

Let  $V(e) = e^T e$ . when  $t = t_k$ , we have

$$\begin{aligned} V(e(t_k^+)) &= (e + B_k e)^T (e + B_k e) \\ &= 0.01e^T e \\ &= 0.01V(e). \end{aligned}$$

When  $t \in (t_k, t_{k+1}]$ ,  $k = 1, 2, \dots$ , we obtain

$$\begin{aligned} \dot{V}(e) &= 2e_1 \dot{e}_1 + 2e_2 \dot{e}_2 + 2e_3 \dot{e}_3 \\ &= 2e_1[a(e_2 - e_1)] + 2e_2[\gamma(|x| - |u|) - (xz - uw) + \beta e_2] \\ &\quad + 2e_3[(xy - uv) - be_3] \\ &\leq 2ae_1 e_2 + 2\gamma|e_1|e_2 - 2ae_1^2 + 2\beta e_2^2 - 2be_3^2 \\ &\quad - 2(xze_2 - uwe_2 - xye_3 + uve_3) \\ &= 2ae_1 e_2 + 2\gamma|e_1|e_2 - 2ae_1^2 + 2\beta e_2^2 - 2be_3^2 \\ &\quad - 2we_1 e_2 + 2ve_1 e_3 \\ &\leq a(e_1^2 + e_2^2) + \gamma(e_1^2 + e_2^2) - 2ae_1^2 + 2\beta e_2^2 \\ &\quad - 2be_3^2 + |w|(e_1^2 + e_2^2) + |v|(e_1^2 + e_3^2) \\ &= (\gamma - a + |v| + |w|)e_1^2 + (\gamma + a + 2\beta + |w|)e_2^2 \\ &\quad (-2b + |v|)e_3^2 \end{aligned}$$

Choose  $\lambda = \max(\gamma - a + |v| + |w|, \gamma + a + 2\beta + |w|, -2b + |v|)$ . Then

$$\dot{V}(e) \leq \lambda V(e).$$

Let  $\Psi_k(V(e)) = 0.01V(e)$  and  $c(V(e)) = V(e)$ . Then

$$\int_{t_k}^{t_{k+1}} \lambda ds + \int_z^{\Psi_k(z)} \frac{ds}{c(s)} = \lambda \Delta t_k - 4.6052,$$

where  $\Delta t_k = t_{k+1} - t_k$ .

Let  $\gamma_k = 0.1$ , then  $\sum_{k=1}^{\infty} \gamma_k = \infty$ . We choose  $\Delta t_k \leq \frac{4.605}{\lambda}$ ,  $k = 0, 1, \dots$ . By Theorem 3.1. in [16], we see that the error system (13) is attractive. Therefore, we obtain the following

**Synchronization criterion:** *The drive system (11) and the response system (12) are synchronized by impulsive driving if*

$$\Delta t_k \leq \frac{4.605}{\lambda}, k = 0, 1, \dots$$

For simulation, we choose  $a = 35$ ,  $\gamma = 5$ ,  $\beta = 27.5$  and  $b = 3$ . Let system (11) and system (12) start respectively from the initial points (2,3,1) and (8,4,8). Then we get  $\lambda = 185$ . We choose  $\Delta t_0 = \Delta t_1 = \dots = 0.01$ ,  $B_1 = B_2 = \dots = -0.9I$ . Then synchronization is observed in the simulation. The phase portraits and state trajectories of the error system (13) are shown in Fig.4. The phase portraits of the response system (12) and the synchronization portraits are shown in Fig.5.

## 6 Conclusions

We have introduced in this paper a new unified chaotic system that connects the Chen’s chaotic system to the classical Lorenz chaotic system through the Lü chaotic system. The continued transition from the Lorenz and the Chen systems is controlled by  $|x|$ , thereby the new attractor is composed by two parts, one part is half a Lorenz attractor, another part is half a Chen attractor. We have shown that the new unified chaotic system has some different qualitative properties from both the Lorenz system and Chen system along with sharing several important qualitative properties. An electronic circuit has also been designed and built to confirm its dynamics of the new chaotic system. Then we have investigated the impulsive synchronization of the new unified chaotic system and established some synchronization criterion.

## References

[1] S. Čelikovský, and G. Chen, “On a generalized Lorenz canonical form of chaotic systems,” *Int. J. Bifurcation and Chaos*, 2002, 12(8), 1789-1812.

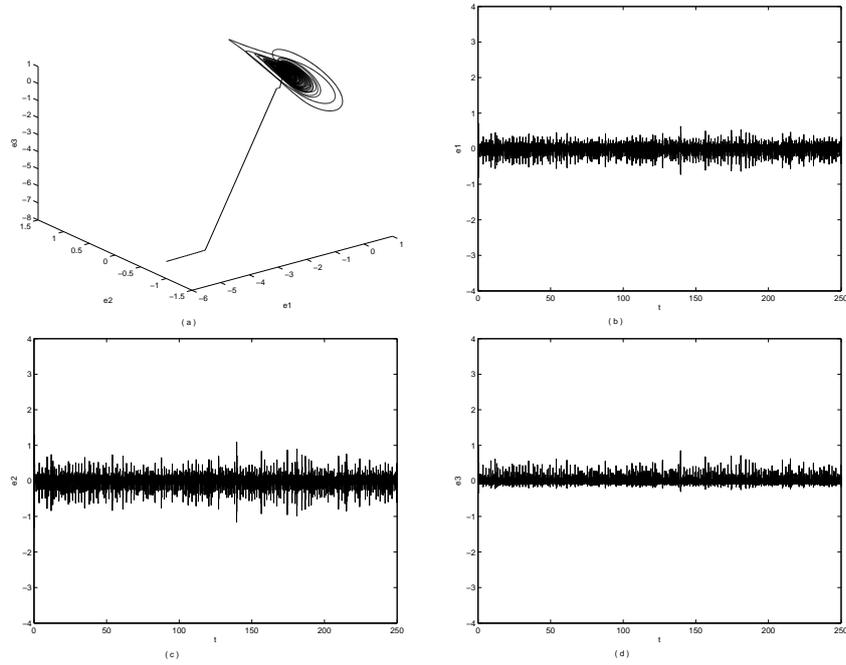


Figure 4: The phase portraits and state trajectories of the error system (13), with  $a = 35$ ,  $\gamma = 5$ ,  $\beta = 27.5$  and  $b = 3$ , starting from the initial points  $(2,3,1)$  and  $(8,4,8)$ : (a)  $e_1 - e_2 - e_3$  plane, (b)  $t - e_1$  plane, (c)  $t - e_2$  plane, (d)  $t - e_3$  plane.

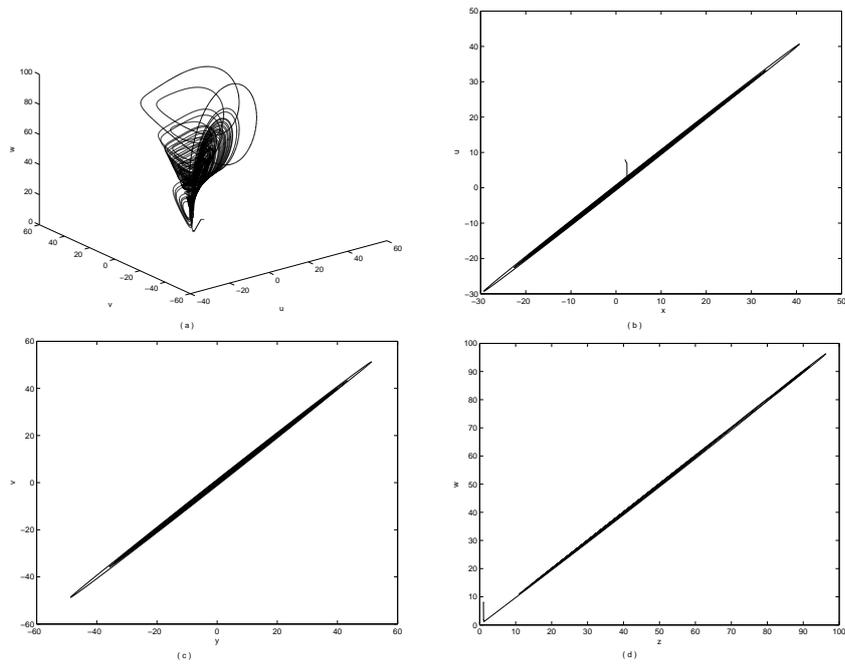


Figure 5: The phase portraits of the driven system (12) and the synchronization portraits: (a)  $u - v - w$  plane, (b)  $x - u$  plane, (c)  $y - v$  plane, (d)  $z - w$  plane.

- [2] G. Chen and T. Ueta, "Yet another chaotic attractor," *Int. J. Bifurcation and Chaos*, 1999, 9(7), 1465-1466.
- [3] K.M. Cuomo and A. Oppenheim, "Chaotic Signals and Systems for Communications", IEEE ICASSP, vol. 3, pp. 137-140, 1993.
- [4] A. Erramilli, R.P. Singh, and P. Pruthi, "Chaotic maps as models of packet traffic," *Proc. 14th Int. Teletraffic Congress*, 1994, 329-338.
- [5] G. Grassi and S. Mascolo, "Non-Linear Observer Design to Synchronize Hyperchaotic Systems via a Scalar Signal", IEEE Trans. Circuits and Sys.-I, vol. 44, pp. 1011-1014, 1997.
- [6] G. Grassi and S. Mascolo, "A System Theory Approach for Designing Cryptosystems Based on Hyperchaos", IEEE Trans. Circuits and Sys.-I, vol. 46, pp. 1135-1138, 1999.
- [7] G. Grassi and S. Mascolo, "Synchronizing Hyperchaotic Systems by Observer Design", IEEE Trans. Circuits and Sys.-II, vol. 46, pp. 478-483, 1999.
- [8] M. Itoh, T. Yang and L.O. Chua, "Experimental Study of Impulsive Synchronization of Chaotic and Hyperchaotic Circuits", *Int. J. Bifur. Chaos*, vol. 9, pp. 1393-1424, 1999.
- [9] M. Itoh, T. Yang and L.O. Chua, "Conditions for Impulsive Synchronization of Chaotic and Hyperchaotic Systems", *Int. J. Bifur. Chaos*, vol. 11, pp. 551-560, 2001.
- [10] M. Itoh, N. Yamamoto, T. Yang and L.O. Chua, "Performance Analysis of Impulsive Synchronization", *Proc. of the 1999 European Conf. on Circuit Theory and Design*, 1999, pp. 353-356.
- [11] A. Khadra, X.Z. Liu and X. Shen, "Application of Impulsive Synchronization to Communication Security", IEEE Trans. Circuits and Sys.-I, vol. 50, pp. 341-351, 2003.
- [12] A. Khadra, X.Z. Liu and X. Shen, "Robust Impulsive Synchronization and Application to Communication Security", *DCDIS, Series B: Applications & Algorithms*, vol. 10, pp. 403-416, 2003.
- [13] L. Kocarev, G.M. Maggio, M. Ogorzalek, L. Pecora, and K. Yao, (eds), "Special issue: Advances on communication systems using chaos," *IEEE Trans. Circuits and Systems I*, 2001, 48(12).
- [14] Z.G. Li, C.Y. Wen and Y.C. Soh, "Analysis and Design of Impulsive Control Systems", IEEE Trans. Automat. Contr., vol. 46, pp. 894-899, 2001.
- [15] Z.G. Li, C.Y. Wen, Y.C. Soh and W.X. Xie, "The Stabilization and Synchronization of Chua's Oscillators via Impulsive Control", IEEE Trans. Circuits and Sys.-I, vol. 48, pp. 1351-1355, 2001.
- [16] X. Liu, "Stability results for impulsive differential systems with applications to population growth models," *Dynamics and Stability of Systems*, 1994, 9, 163-174.
- [17] E. N. Lorenz, "Deterministic nonperiodic flows," *J. Atmos. Sci.*, 1963, 20, 130-141.
- [18] J. Lü, and G. Chen, "A new chaotic attractor coined," *Int. J. Bifurcation and Chaos*, 2002, 12(3), 659-661.
- [19] J. Lü, G. Chen, S. Zhang, and S. Čelikovský, "Bridge the gap between the Lorenz system and the Chen system," *Int. J. Bifurcation and Chaos*, 2002, 12(12), 2917-2926.
- [20] H. Nijmeijer and I.M.Y. Mareels, "An Observer Looks at Synchronization", IEEE Trans. Circuits and Sys.-I, vol. 44, pp. 882-890, 1997.
- [21] L. Pecora and T. Carroll, "Synchronization in Chaotic Systems", *Phys. Rev. Lett.*, vol. 64, pp. 821-824, 1990.

- [22] J.H. Peng, E.J. Ding, M. Ding and W. Yang, "Synchronizing Hyperchaos with Scalar Transmitted Signal", *Phys. Rev. Lett.*, vol. 76, pp. 904-907, 1996.
- [23] C. P. Silva, "Shil'nikov's theorem - A tutorial," *IEEE Trans. Circuits and Syst.*, 1993, 40, 675-682.
- [24] H. Sira-Ramírez and C. Cruz-Hernández, "Synchronization of Chaotic Systems: A Generalized Hamiltonian Systems Approach", *Int. J. Bifur. Chaos*, 11, pp. 1381-1395, 2001.
- [25] T. Stojanovski, L. Kocarev and U. Parlitz, "Driving and Synchronizing by Chaotic Impulses", *Phys. Rev. E*, vol. 43, pp. 782-785, 1996.
- [26] J.A. Suykens, P.F. Curran and L.O. Chua, "Robust Synthesis for Master-Slave Synchronization of Lur'e Systems", *IEEE Trans. Circuits and Sys.-I*, vol. 46, pp. 841-850, 1999.
- [27] J.A. Suykens, P.F. Curran, J. Vandewalle and L.O. Chua, "Robust Non-Linear  $H_\infty$  Synchronization of Chaotic Lur'e Systems", *IEEE Trans. Circuits and Sys.-I*, special issue on Chaos Synchronization, Control and Applications, vol. 44, pp. 891-904, 1997.
- [28] J.A. Suykens, T. Yang and L.O. Chua, "Impulsive Synchronization of Chaotic Lur'e Systems by Measurement Feedback", *Int. J. Bifur. Chaos*, vol. 8, pp. 1371-1381, 1998.
- [29] T. Yang and L.O. Chua, "Impulsive Stabilization for Control and Synchronization of Chaotic Systems: Theory and Application to Secure Communication", *IEEE Trans. Circuits and Sys.-I*, vol. 44, pp. 976-988, 1997.
- [30] T. Yang and L.O. Chua, "Impulsive Control and Synchronization of Non-Linear Dynamical Systems and Application to Secure Communication", *Int. J. Bifur. Chaos*, vol. 7, pp. 645-664, 1997.
- [31] T. Yang and L.O. Chua, "Generalized Synchronization of Chaos via Linear Transformations", *Int. J. Bifur. Chaos*, vol. 9, pp. 215-219, 1999.

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